A STABILITY ANALYSES OF SYMMETRIC 2-D BUOYANT HEAT PLUME FLOW

Kou Xiaojun and Wang Qingan
(State Key Laboratory of Fire Science, Hefei, Anhui, 230026, China)

ABSTRACT

In this paper, we will deduce a criterion of stability of symmetric two-dimensional buoyant heat plume flow using perturbation method.[1] We will also get some result about the stability of the flow, which the flow' characteristic and the wave length of the perturbation wave effect on, and analyze the mechanism of instability of the flow physically.

KEY WORDS: Buoyant Flow, Stability Theory,
Perturbation Method

INTRODUCTION

The authors notice the problem of stability of buoyant heat plume flow during numeric simulation of the flow.[2] When calculating two-dimensional buoyant heat plume flow with symmetric boundary conditions, we find the calculated result is not symmetric. Is that caused by instability of the buoyant heat plume flow? By analyses and theoretic deduction, we get the conclusion that there do exist the possibility of instability in buoyant heat plume flow, gain a criterion of stability of symmetric two-dimensional buoyant heat plume flow, and obtain some primary result about the stability of the flow.

MATHEMATIC DEDUCTION

The control equation group of the two-dimensional buoyant plum flow is:

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho w)}{\partial z}=0$$

$$\frac{\partial(\rho u)}{\partial t}+\frac{\partial(\rho uu)}{\partial x}+\frac{\partial(\rho uw)}{\partial z}=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial(\rho w)}{\partial t}+\frac{\partial(\rho uw)}{\partial x}+\frac{\partial(\rho ww)}{\partial z}=-\frac{\partial p}{\partial z}+\mu\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial z^2}+(\rho_0-\rho)g$$

$$\frac{\partial(\rho T)}{\partial t}+\frac{\partial(\rho uT)}{\partial x}+\frac{\partial(\rho wT)}{\partial z}=\frac{\mu}{\sigma_h}\frac{\partial^2 T}{\partial x^2}+\frac{\mu}{\sigma_h}\frac{\partial^2 T}{\partial z^2}$$

where x axis is horizontal axis, z axis is vertical axis.

When the change of temperature of heat plume flow is not very great, we can take the change of density into consideration only in the the item of buoyancy, and do not take the change of density into consideration in other items. Let the equations above be divided by ρ_0, the equations change into:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \gamma\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \gamma\frac{\partial^2 w}{\partial x^2} + \gamma\frac{\partial^2 w}{\partial z^2} + \frac{T-To}{To}g$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \frac{\gamma}{\sigma_h}\frac{\partial^2 T}{\partial x^2} + \frac{\gamma}{\sigma_h}\frac{\partial^2 T}{\partial z^2}$$

The equations are linked each other and non-linear, and they can not be solved by analytic method. We linearize them using perturbation method. The main ideas of the method are:

A. The variables of the flow can be divided into average variables and perturbation variables.

B. The average variables can fit to the original equation group.

C. The perturbation variables are relatively small to the average variables, so we can omit the two-power items of perturbation variables

Let $u=\bar{u}+u'$, $w=\bar{w}+w'$, $p=\bar{p}+p'$, $T=\bar{T}+T'$, substituting this to the original group, and omitting the two-power items of the perturbation variables, we get:

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \bar{w}\frac{\partial u'}{\partial z} + u'\frac{\partial \bar{u}}{\partial x} + w'\frac{\partial \bar{u}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \gamma\frac{\partial^2 u'}{\partial x^2} + \gamma\frac{\partial^2 u'}{\partial z^2}$$

$$\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x} + \bar{w}\frac{\partial w'}{\partial z} + u'\frac{\partial \bar{w}}{\partial x} + w'\frac{\partial \bar{w}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{T'}{To}g$$

$$+\gamma \frac{\partial^2 w'}{\partial x^2} + \gamma \frac{\partial^2 w'}{\partial z^2}$$

$$\frac{\partial T'}{\partial t} + \bar{u}\frac{\partial T'}{\partial x} + \bar{w}\frac{\partial T'}{\partial z} + u'\frac{\partial \bar{T}}{\partial x} + w'\frac{\partial \bar{T}}{\partial z} = \frac{\gamma}{\sigma_h}\frac{\partial^2 T'}{\partial x^2} + \frac{\gamma}{\sigma_h}\frac{\partial^2 T'}{\partial z^2}$$

The main direction of the flow is z ($\bar{u} \ll \bar{w}$), so we can omit $\bar{u}\frac{\partial u'}{\partial x}$, $\bar{u}\frac{\partial w'}{\partial x}$, $\bar{u}\frac{\partial T'}{\partial x}$. We know $\frac{\partial \bar{u}}{\partial z}$ $\frac{\partial \bar{w}}{\partial x}$ $\frac{\partial \bar{T}}{\partial z}$ is relatively small in two dimentional buoyant plume flow, so we omit the items containing them. the equation group changes into:

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial u'}{\partial t} + \bar{w}\frac{\partial u'}{\partial z} + u'\frac{\partial \bar{u}}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \gamma\frac{\partial^2 u'}{\partial x^2} + \gamma\frac{\partial^2 u'}{\partial z^2}$$

$$\frac{\partial w'}{\partial t} + \bar{w}\frac{\partial w'}{\partial z} + w'\frac{\partial \bar{w}}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \gamma\frac{\partial^2 w'}{\partial x^2} + \gamma\frac{\partial^2 w'}{\partial z^2} + \frac{T'}{To} g$$

$$\frac{\partial T'}{\partial t} + \bar{w}\frac{\partial T'}{\partial z} + u'\frac{\partial \bar{T}}{\partial x} = \frac{\gamma}{\sigma_h}\frac{\partial^2 T'}{\partial x^2} + \frac{\gamma}{\sigma_h}\frac{\partial^2 T'}{\partial z^2}$$

Next we discuss this eqation $\frac{\partial T'}{\partial t} + \bar{w}\frac{\partial T'}{\partial x} + u'\frac{\partial \bar{T}}{\partial z} = \frac{\gamma}{\sigma_h}\frac{\partial^2 T'}{\partial x^2} + \frac{\gamma}{\sigma_h}\frac{\partial^2 T'}{\partial z^2}$.

$\bar{w}\frac{\partial T'}{\partial z}$ is the convection of average vertical velocity to perturbation temperature.

$\frac{\gamma}{\sigma_h}\frac{\partial^2 T'}{\partial x^2} + \frac{\gamma}{\sigma_h}\frac{\partial^2 T'}{\partial z^2}$ is the diffusion item of perturbation temperature. In order

order to be simple, we do not consider these items. the equation of temperature changes into:

$$\frac{\partial T'}{\partial t} + u'\frac{\partial \bar{T}}{\partial x} = 0 \, .$$

That means the perturbation temperature is caused only by the convection of horizontal perturbation velocity to the average temperature. The final simplified equation is:

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$$

$$\frac{\partial u'}{\partial t} + \bar{w}\frac{\partial u'}{\partial z} + u'\frac{\partial \bar{u}}{\partial x} = - \frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \gamma \frac{\partial^2 u'}{\partial x^2} + \gamma \frac{\partial^2 u'}{\partial z^2}$$

$$\frac{\partial w'}{\partial t} + \bar{w}\frac{\partial w'}{\partial z} + w'\frac{\partial \bar{w}}{\partial z} = - \frac{1}{\rho_0}\frac{\partial p'}{\partial z} + \frac{T'}{To} g + \gamma \frac{\partial^2 w'}{\partial x^2} + \gamma\frac{\partial^2 w'}{\partial z^2}$$

$$\frac{\partial T'}{\partial t} + u'\frac{\partial \bar{T}}{\partial x} = 0$$

By eliminating to the perturbation variables, using some simplifying methods mentioned above, and omitting the items which contain derivatives of average variables higher then second order, we get the differential equation of single perturbation variable. It is:

$$\Big[\frac{\partial^4}{\partial t^2\,\partial z^2} + \frac{\partial^4}{\partial t^2\,\partial x^2} + \frac{\partial \bar{w}}{\partial z}\,\frac{\partial^3}{\partial t\,\partial z^2} + 2\frac{\partial \bar{w}}{\partial z}\,\frac{\partial^3}{\partial t\,\partial x^2}$$

$$+ \bar{w}\frac{\partial^4}{\partial t\,\partial z^3} + \bar{w}\frac{\partial^4}{\partial t\,\partial x^2\,\partial z} - \frac{g}{To}\,\frac{\partial^2 \bar{T}}{\partial x^2}\,\frac{\partial}{\partial z}$$

$$- \frac{g}{To}\,\frac{\partial \bar{T}}{\partial x}\,\frac{\partial^2}{\partial x\,\partial z} - \gamma\frac{\partial^5}{\partial t\,\partial x^4} - 2\gamma\frac{\partial^5}{\partial t\,\partial x^2\,\partial z^2}$$

$$- \gamma\frac{\partial^5}{\partial t\,\partial z^4}\Big] u = 0$$

Assuming the perturbation variable has the below form:

$$u = U\, e^{i(kx+nz-wt)}$$
$$= U\, e^{bt}\, e^{i(kx+nz-at)}$$

here $\omega=a+bi$

(If $b>0$, the amplitude of wave increases with time, the flow is instable. If $b<0$ or $b=0$, the amplitude of wave decrease with time or does not change, the flow is stable.)

Substituting this form to equation group, we get the the frequency equation as follow:

$$\omega^2 + (A + Bi)\omega + C + Di = 0$$

$$A = -n\bar{w}$$

$$B = \frac{\partial \bar{w}}{\partial z}\left[\frac{n^2 + 2k^2}{n^2 + k^2}\right] + (n^2 + k^2)\gamma$$

$$C = \frac{g}{To}\frac{\partial \bar{T}}{\partial x}\frac{nk}{n^2 + k^2}$$

$$D = -\frac{g}{To}\frac{\partial^2 \bar{T}}{\partial x^2}\frac{n}{n^2 + k^2}$$

Substituting w=a+bi to the frequency equation, we get the stability criterion (b<0 or b=0) as follow:

$$|D| < \left[\frac{|A|}{2} + \sqrt{A^2 - 4C}\right]|B|$$

CONCLUSION

1. Relationship Between Stability Of Flow And Average Flow

A. The bigger the $|D|$ is, the the easier for the flow to be instable. That is to say the sharper the temperature curve is, the easier for the flow to be instable.

B. The bigger the $|A|$ is, the uneasier for the flow to be instable. That is to say the higher upward velocity is, the uneasier for the flow to be instable.

C. The bigger the $|B|$ is, the uneasier for the flow to be instable. That is to say, the greater the acceleration of upward velocity is, the the uneasier for the flow to be in stable, and the larger the viscosity is (when the flow is turbulent,the viscosity is effective viscosity), the uneasier for the flow to be instable.

D. If C<0, the flow is easier to be instable, C>0 the flow is uneasier to be instable. That is to say, when the direction of perturbation wave is the same as that of the temperature gradient, the flow is easier to be instable, when it is opposed to thatof the temperature gradient, the flow is uneasier to be instable. That also means when the direction of perturbation wave is the same as that of the temperature gradient, it is easier to grow, when it is opposed to that of the temperature gradient, it is uneasier to grow.

2. Relationship Between Stability And Wavelength Of Perturbation Wave

A. The bigger the $|B|$ is, the easier for the flow to be instable. That means when the average flow does not change, the longer the wave length is, the easier for the flow to be instable.

B. Generally, to the small-scaled flow, the temperature changes sharply, that is advantageous factor for the flow to be instable, but the effect of the viscosity increases more rapidly, that is disadvantageous factor for the flow to be instable. so, we can see, if the scale of the flow is too small, the flow is uneasier to be instable. Whereas, if the scale of the flow is too large, the change of the temperature is too smooth for flow to be instable. Only when the flow is within a certain scale, it is easiest for the flow to be instable.

DISCUSSION AND PHYSICAL ANALYSES

It must be pointed out that the criterion of the stability of buoyant flow given above is not accurate enough to be used in real calculation because there had to much simplifying in deduction. Our intention is to get the tendency of stability of the heat plume·flow, on which the the characteristic of average flow and perturbation wave effect, so the criterion given above is enough. If you want a more accurate criterion for calculation, you should do more complicated deduction, and there should be more items in the criterion.

The buoyant heat plume flow as show in figure 1, if there is a perturbation velocity to lift at A, to right at B, the crest of the temperature will slant to lift at A, to right at B. So the buoyancy at lift side of A will bigger than average. It will induces a upward perturbation velocity at C, and a downward velocity at D to supply the mass loss at the right side of A. Thus, we can see, there exist a circulation of ACBD. The buoyancy of the perturbation temperature is the impetus of circulation. If the temperature is sharp enough to sustain or grow the circulation, but not let the circulation be 'blown out' or be dissipated out by internal friction, the flow is instable. Otherwise, it is stable.

It has been pointed out by Donald A.Haines[3] that the buoyant heat plume flow above intense wild land fire, in some occasion, is not symmetric, but is what shown in figure 2. According our theory mentioned above, to the buoyant heat plume flow above intense wild land fire, the perturbation can grow strongly and make the flow to be non-symmetric, that may be the mechanism of horizontal fire tornado.

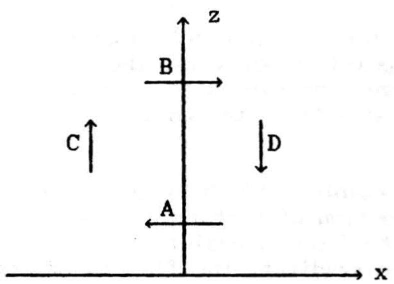

Fig 1.

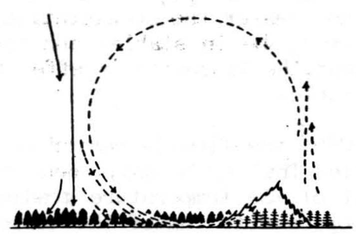

Fig 2.

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